Character tables of finite groups

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Providence, Nov 1st, 2018

Character tables

G finite group

$\mathrm{Irr}(G) = $ trace functions of complex irreducible matrix representations

$$G \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

$\Big(\chi(g)\Big)_{\chi \in \mathrm{Irr}(G), g \in G/\sim}$ *character table of G*

Encodes important information on G.

Frobenius (≈ 1900):

- $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$,   degree $\chi(1)$ divides $|G|$
- $g \in G$ is commutator of two elements $\Longleftrightarrow \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \neq 0$.
- $C, D \subset G$ conjugacy classes;
  g is product of some $c \in C, d \in D \Longleftrightarrow \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(c)\chi(d)\overline{\chi(g)}}{\chi(1)} \neq 0$

Reduction approach

Mathematical problems involving symmetries can often be reduced to questions on finite groups.

Problems on finite groups can often be reduced to questions on (nearly) simple groups (e.g., O'Nan–Scott theorem)

Questions about (nearly) simple groups can often be solved using knowledge of their character tables.

Examples:

- construction of Galois extensions with given group
- monodromy groups of Riemann surfaces
- existence of Beauville surfaces
- local-global conjectures in representation theory

The classification of finite simple groups (CFSG)

A non-abelian finite simple group is one of:

- sporadic simple group, e.g., the Mathieu groups, the Monster
- alternating group
- group of Lie type, e.g., $\mathrm{PSL}_n(\mathbb{F}_q)$, $E_8(\mathbb{F}_q)$, ${}^3D_4(q)$

Character tables described by

- $\mathbb{ATLAS}$ for sporadic simple groups
- combinatorial methods for alternating groups
- Lusztig's geometric approach for groups of Lie type

The ATLAS of finite groups

J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson (1985):

> *Atlas of finite groups*

Collects 399 character tables of (nearly) simple groups:
automorphism groups, Schur covers, ...

Until now: **1883** citations in *MathSciNet*

Jean-Pierre Serre "cannot think of any other book published in the past 50 years which had such an impact".

But: No proofs!

April 24, 2015: in talk at Harvard, J.-P. Serre asks for a verification of information in ATLAS .

The verification project

Wrong character values: known in only **three** $\mathbb{ATLAS}$ tables
(not simple, but "almost simple" groups). Errata available on webpage.
Tables in computer algebra systems GAP, Magma.

Thomas Breuer, G.M., Eammon O'Brien (2015):

Independent verification of $\mathbb{ATLAS}$ tables.

For every group G occurring in the $\mathbb{ATLAS}$

(1) find some realisation $\tilde{G}$ of G (by matrices, permutations);
(2) compute character table of $\tilde{G}$ (algorithm of Unger);
(3) verify that the computed table must be the table of G (with CFSG).

Unger's algorithm (2006)

Based on:

Brauer's Induction Theorem

Any $\chi \in \text{Irr}(G)$ is $\mathbb{Z}$-linear combination of characters induced from elementary subgroups (that is, p-group–by–cyclic).

(originally used to show that *L*-functions are meromorphic)

Algorithm: • induce all characters from all elementary subgroups
• $\text{Irr}(G)$ by LLL-reduction of this lattice of class functions

Theorem (Breuer-M.-O'Brien, 2017)

The character tables of all $\mathbb{ATLAS}$ groups except for B and M have been recomputed. No further errors were found.

April 28, 2017: in talk at Harvard, J.-P. Serre acknowledges his concerns have been met.

Groups of Lie type

Best studied starting from linear algebraic groups:

$\mathbf{G}$ simple algebraic group of simply connected type over $\overline{\mathbb{F}}_q$,
$F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism,
$G = \mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$ finite group of fixed points.

Then $G/Z(G)$ is simple; all simple groups of Lie type obtained this way.

Example ($\mathbf{G} = \mathrm{SL}_n$)

(1) If $F : \mathrm{SL}_n \to \mathrm{SL}_n$: $(a_{ij}) \mapsto (a_{ij}^q)$, then
$G = \mathbf{G}^F = \mathrm{SL}_n(\mathbb{F}_q)$ special linear group and $G/Z(G) = \mathrm{PSL}_n(q)$.

(2) If $F : \mathrm{SL}_n \to \mathrm{SL}_n$: $(a_{ij}) \mapsto ((a_{ij}^q)^{\mathrm{tr}})^{-1}$, then
$G = \mathbf{G}^F = \mathrm{SU}_n(q)$ special unitary group and $G/Z(G) = \mathrm{PSU}_n(q)$.

Characters of G: construction from algebraic group.

Lusztig's theory

Let $\mathbf{L} \leq \mathbf{G}$ be a Levi subgroup, stable under F.

Define *Deligne–Lusztig varieties* X from $\mathbf{G}$, with action of $\mathbf{G}^F \times \mathbf{L}^F$

$\Longrightarrow$ ℓ-adic cohomology groups $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ are $\mathbf{G}^F \times \mathbf{L}^F$-bimodules.

This defines *Lusztig induction* $R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F) \longrightarrow \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F)$.

Theorem (Lusztig, 1984/1988)

The irreducible characters of $\mathbf{G}^F$ can be parametrised explicitly in terms of suitable combinatorial objects.

Parameters: • semisimple element s in *dual* group $\mathbf{G}^{*F}$
- *unipotent character* ψ of $C_{\mathbf{G}^{*F}}(s)$

Jordan decomposition of characters

Lusztig's parametrisation:

$$\text{Irr}(\mathbf{G}^F) = \bigsqcup_s \mathcal{E}(\mathbf{G}^F, s) \quad \text{(Lusztig series)}$$

and

$$J : \mathcal{E}(\mathbf{G}^F, s) \xrightarrow{1-1} \mathcal{E}(C_{\mathbf{G}^{*F}}(s), 1) \quad \text{(unipotent characters)}$$

with

$$\chi(1) = |\mathbf{G}^{*F} : C_{\mathbf{G}^{*F}}(s)|_{p'} \, J(\chi)(1) \quad \text{(degree formula)}.$$

In particular, degrees of all character are known.

But: **not** the character values on all classes!

Generic character tables

Want to work with the character tables of a whole family of groups of Lie type at the same time, on a computer, e.g., for $SL_2(q)$ for all $q = p^f$.

Leads to concept of *generic character tables*.

Generic character table of $SL_2(q)$, $q = 2^f$

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$S(a)$	$T(b)$
1_G	1	1	1	1
St_G	q	0	1	-1
$\chi^{(i)}$	$q+1$	1	$\varepsilon^{ai} + \varepsilon^{-ai}$	0
$\psi^{(j)}$	$q-1$	-1	0	$-\eta^{bj} - \eta^{-bj}$

$1 \le i, a \le \frac{q}{2} - 1$, $1 \le j, b \le \frac{q}{2}$

Here, $\varepsilon = \exp(2\pi i/(q-1))$, $\eta = \exp(2\pi i/(q+1))$.

The Chevie project

Generic character table for a family $\{G(q) \mid q = p^f\}$:

- one column for all classes $C = [g]$ with conjugate centralisers $C_G(g)$
- one row for all characters $\chi = \chi_{s,\psi}$ with conjugate centralisers $C_{G^*}(s)$

GAP package **Chevie** (started in 1996 by M. Geck, G. Hiß, F. Lübeck, G. M., G. Pfeiffer, now mainly developed by F. Lübeck and J. Michel): generic character tables, e.g. for

$$\mathrm{SL}_3(q),\ \mathrm{PGL}_3(q),\ \mathrm{SU}_3(q),\ \mathrm{Sp}_4(q),\ G_2(q),\ {}^3D_4(q), \ldots$$

and functionality like

- scalar products of characters
- tensor products of characters
- class multiplication constants
- induction from certain subgroups
- computations in Weyl groups and Hecke algebras

168 citations in *MathSciNet*; is used by Lusztig

Constructing new generic tables

Also a tool to determine new tables, e.g., for

$$SL_4(q), \ Spin_8^+(q), \ Spin_8^-(q), \ F_4(q) \qquad \text{(work in progress)}$$

for which Lusztig's theory does not give all values.

Example

Generic table for $Spin_8^+(q)$ will have 237 columns and 579 rows, entries are polynomials in q with coefficients generic roots of unity.

Long term goal:

$$\text{generic table of } E_8(q) \ (\approx 6000 \text{ rows and columns});$$

will probably require to first treat $D_4(q) = Spin_8^+(q)$, $D_5(q)$, $E_6(q)$, $E_7(q)$.

At present: Explicit computations seem only way to overcome missing theory for character values.

Applications 1: Galois realisations

Rigidity criterion (Belyi, Fried, Matzat, Thompson):

G with trivial centre is Galois group of some extension $N/\mathbb{Q}^{\mathrm{ab}}$ if there are $x, y \in G$ with

- $$G = \langle x, y \rangle$$

- $$\frac{|G|}{|C_G(x)| \cdot |C_G(y)| \cdot |C_G(xy)|} \sum_{\chi \in \mathsf{Irr}(G)} \frac{\chi(x)\chi(y)\chi(xy)}{\chi(1)} = 1$$

Theorem (Guralnick–M., 2014)

The groups $E_8(\mathbb{F}_p)$, $p \geq 7$, occur as Galois groups over $\mathbb{Q}$.

Uses Chevie calculations and estimates on character values.

Applications 2: Simple groups

Ore's conjecture

G finite simple $\implies$ every $g \in G$ is a commutator.

Proved by Liebeck, O'Brien, Shalev, Tiep (2010), also using explicitly constructed character tables, and estimates on character values.

Thompson's conjecture

G finite simple $\implies$ there is a class $C \subset G$ with $C \cdot C = G$.

Open. Exceptional type groups could be solved with generic tables.

Arad–Herzog conjecture

G finite simple, $C, D \subset G \setminus \{1\}$ classes $\implies C \cdot D$ never is a single class.

Open. Again, exceptional type groups could be solved with generic tables.

Applications 3: Local-global conjectures

ℓ a prime number

$\mathrm{Irr}_{\ell'}(G) := \{\chi \in \mathrm{Irr}(G) \mid \chi(1) \text{ prime to } \ell\}$

Conjecture (McKay, 1972)

G a finite group, $P \in \mathrm{Syl}_\ell(G)$. Then

$$|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(N_G(P))|.$$

Isaacs–M.–Navarro (2007): reduced to (difficult) condition for simple G.

Theorem (M.–Späth, 2016)

The McKay-conjecture holds for the prime $\ell = 2$.

$\ell > 2$: open questions on action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$.

Applications 4: Brauer blocks

To understand ℓ-modular representations

$$G \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_\ell),$$

determine *Brauer ℓ-blocks* of G.

Using Lusztig's results, work of many authors gives: ℓ-blocks of groups of Lie type known, except for exceptional groups at primes $\ell \leq 5$.

Strong compatibility with Lusztig series, e.g., Broué–Michel:

$$\mathcal{E}_\ell(G, s) = \bigsqcup_t \mathcal{E}(G, st) \qquad (t \in C_{G^*}(s)_\ell)$$

is a union of ℓ-blocks.

Combinatorial description in terms of Lusztig induction $R_{\mathbf{L}}^{\mathbf{G}}$.

Applications 5: More local-global conjectures

Brauer: ℓ-block B of G has *defect group* $D \leq G$ (an ℓ-subgroup)

Conjecture (Brauer, 1955)

G a finite group, B an ℓ-block with defect group D. Then

$$\chi(1)_\ell = |G : D|_\ell \text{ for all } \chi \in \text{Irr}(B) \Longleftrightarrow D \text{ abelian}.$$

Berger–Knörr (1988): "$\Longleftarrow$"-direction reduced to quasi-simple groups.

Theorem (Kessar–M., 2013)

The "$\Longleftarrow$"-direction of Brauer's conjecture holds.

Navarro–Späth (2012): "$\Longrightarrow$"-direction reduced to (difficult) condition for simple groups. Open.

Applications 6: More conjectures

B an ℓ-block with defect group D.
Height $\text{ht}(\chi)$ of $\chi \in \text{Irr}(B)$ defined by

$$\chi(1)_\ell = \ell^{\text{ht}(\chi)} |G : D|_\ell.$$

Conjecture (Robinson, 1995)

For all $\chi \in \text{Irr}(B)$,

$$\ell^{\text{ht}(\chi)} \leq |D : Z(D)|.$$

Murai (1996): reduced to quasi-simple groups.

Theorem (Feng–Li–Liu–M.–Zhang, 2018)

Robinson's conjecture holds for $\ell > 2$.

Small/exceptional cases: use known character tables.

Open questions

Fundamental open problem

Determine the (degrees of the) ℓ-modular Brauer characters of the (nearly) simple groups.

Groups of Lie type in characteristic ℓ: Lusztig's conjecture.

Groups of Lie type in characteristic $p \neq \ell$, sporadic groups: determine *decomposition matrices*.

Heavy use of computational methods and tools.